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Construction for a Class of Borderenergetic Digraphs

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Abstract: The energy of a digraph is defined as the sum of all real parts of its eigenvalues which are respect to its adjacency matrix. It is well known that graph energy is found that there are many applications in chemistry, physics and biology. In 2015, Gong and Gutman et al. proposed the concept of a *borderenergetic* graph. That is, if a graph G of order n satisfies its graph energy is equal to the value obtained by using twice of its order minus two, then G is called a borderenergetic graph. That is, the energies of borderenergetic graphs are equal to those of complete graphs of the same orders. Note that a graph is also a special digraph. Naturally, the concept of a borderenergetic digraph is extended to digraph energy. In this work, we first characterize its matrix and obtain the relationship between the spectra of a digraph and its complement. By using the spectra of the complete product between two regular digraphs, a kind of borderenergetic digraphs can be constructed. Furthermore, based on the results before, a class of sequences of borderenergetic digraphs can be constructed.

Keywords: Regular Digraphs, Spectrum of a Digraph, Digraph Energy, Borderenergetic Digraphs

1. Introduction

Let $D = (V(D), A(D))$ be a digraph with the vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and the arc set $A(D) = \{e_1, e_2, \cdots, e_m\}$ consisted of ordered pairs of distinct vertices. If $e_l = (v_i, v_j)$ is an arc of D, then v_i is said to be adjacent to v_j. The number of arcs starting from v_i is called the outdegree of the vertex v_i denoted by $od(v_i)$. Similarly, the number of arcs ending at v_i is called the indegree of the vertex v_i denoted by $id(v_i)$. A digraph D with n vertices is r-regular if $od(v_i) = id(v_i) = r$ for $i = 1,2,\cdots n$. A digraph D is strongly connected if for every two distinct vertices v_i and v_j, there exists two walks from v_i to v_j and from v_j to v_i $(i \neq j)$. The complete graph K_n is the graph with exactly one edge between every two different vertices. Similarly, the symmetric complete digraph is denoted by $\overleftrightarrow{K_n}$ and every edge in K_n is replaced by a pair of symmetric arcs in $\overleftrightarrow{K_n}$. The complement $\overline{D}$ of a digraph D is the digraph with the set of vertices $V(\overline{D}) = V(D)$, and if v_i and v_j are two distinct vertices of $\overline{D}$ and $(v_i, v_j) \in E(\overline{D})$, then $(v_i, v_j) \notin E(D)$. In Figure 1, the complement $\overline{D}$ of digraph D is shown. We see that $V(\overline{D}) = V(D)$ and $E(\overline{D}) \cup E(D) = E(\overleftrightarrow{K_n})$.

In chemistry, the well-known Hückel molecular orbital (HMO) theory [15] is often used to estimate the energy of π-electrons orbital in conjugated hydrocarbon molecules. Later, chemists realized that the matrix used by Hückel was the adjacency matrix of the graph related to a molecular structure. Based on some reasonable assumptions, the total energy of π-electrons can be written as the sum of the absolute values of the eigenvalues of the graph. That is graph energy.

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of its adjacency matrix $A(G)$ for a graph G. The energy of graph $G[12]$ is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For graph energy, there are many applications and results can be seen in the references [3, 10, 13-14, 17, 21].

Recently, the concept of a *borderenergetic* graph is proposed by Gong and Gutman et al [9]. If a graph G of order n satisfies $E(G) = 2(n-1)$, then G is called a borderenergetic graph. More results on borderenergetic graphs can be found in the references [4-5, 7-8, 16].

On the other hand, the concept of energy was extended to the digraphs by Rada [22]. If D is a digraph with the eigenvalues $z_1, z_2, \cdots, z_n$ for its adjacency matrix $A(D)$, then the energy of digraph D is defined as

$$E(D) = \sum_{i=1}^{n} |Rez_i|.$$

For more results on the energies and spectra of digraphs, one refers to the references [1-2, 6, 11, 18-20, 23-25]. Note that a graph is also a special digraph. Naturally, the concept of a borderenergetic graph can be extended to a digraph.

If a digraph D of order n satisfies $E(D) = 2(n-1)$, then D is called a *borderenergetic digraph*. In this work, by using the spectra for the complement of a regular digraph and the complete product between two regular digraphs, a class of sequences of borderenergetic digraphs are constructed.

2. Spectrum of Complete Product of Digraphs

For the operation $D_1 \oplus D_2$, its vertex set is $V(D_1 \oplus D_2) = V(D_1) \cup V(D_2)$ and its arc set is $E(D_1 \oplus D_2) = E(D_1) \cup E(D_2)$. The complete product $D_1 \nabla D_2$ of the digraphs D_1 and D_2 is obtained from each vertex in D_1 joins with each vertex in D_2 by a pair of symmetric arcs. An example of complete product is given in Figure 2. It is easy to see that $V(D_1 \nabla D_2) = V(D_1) \cup V(D_2)$ and $E(D_1 \nabla D_2) = E(D_1) \cup E(D_2) \cup \{(x,y),(y,x) | x \in V(D_1), y \in V(D_2)\}$.

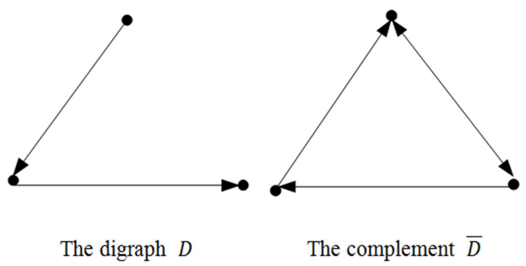


Figure 1. *The digraph D and its complement $\overline{D}$.*

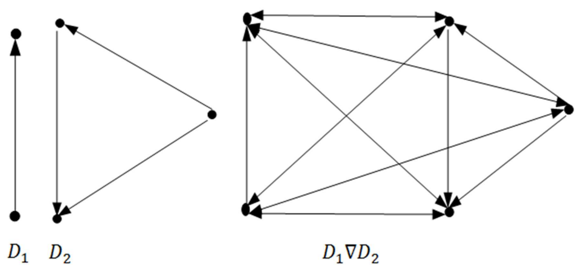


Figure 2. *The digraphs D_1, D_2 and $D_1 \nabla D_2$.*

Lemma 1. Let D be a strongly connected r-regular digraph of order n. Suppose $z_1, z_2, \cdots z_n$ are the eigenvalues of D. If $r = z_1 > Rez_2 \geq \cdots \geq Rez_n$, then the eigenvalues of $\overline{D}$ are $n-1-r$, $-1-z_2$, $-1-z_3$, $\cdots$, $-1-z_n$.

Proof The adjacency matrix of the complement of D is denoted by $A(\overline{D}) = J - I - A(D)$, where I is an identity matrix and J is the matrix with each of whose entries is equal to 1. Since D is r-regular, we have $z_1 = r$ with corresponding eigenvector $\alpha = \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix}$. Let x_2, x_3, $\cdots$, x_n be the eigenvectors of $A(D)$ corresponding to the eigenvalues $z_2, z_3, \cdots, z_n$. Then

$$A(D)x_i = z_i x_i,\ i = 2,3,\cdots,n.$$

$$\alpha^T A(D) x_i = \alpha^T z_i x_i,$$

$$r\alpha^T x_i = z_i \alpha^T x_i.$$

For $i = 2,3,\cdots,n$, $r \neq z_i$, $\alpha^T x_i = 0$, $i = 2,3,\cdots,n$.

Since J can be expressed as $J = \begin{pmatrix} \alpha^T \\ \alpha^T \\ \vdots \\ \alpha^T \end{pmatrix}$, we get $Jx_i = 0$, $i = 2,3,\cdots,n$. Hence,

$$A(\overline{D})x_i = [J - I - A(D)]x_i = Jx_i - x_i - A(D)x_i = (-1 - z_i)x_i,\ i = 2,3,\cdots,n.$$

Therefore, $(-1-z_i)$ is an eigenvalue with corresponding eigenvector x_i of $A(\overline{D})$, where $i = 2,3,\cdots,n$. As $\overline{D}$ is a $(n-1-r)$-regular digraph of order n, we see that $(n-1-r)$ is an eigenvalue $A(\overline{D})$. □

By using lemma 1, we can obtain the spectrum of the complete product for two regular digraphs.

Theorem 2. Let D_1 be a strongly connected r_1-regular digraph of order n_1 and D_2 be a strongly connected r_2-regular digraph of order n_2. Suppose $r_1 + n_2 = r_2 + n_1$. Let $z_1, z_2, \cdots z_{n_1}$ and $z_1', z_2', \cdots z_{n_2}'$ be the eigenvalues of D_1 and D_2, respectively. If $r_1 = z_1 > Rez_2 \geq \cdots \geq Rez_{n_1}$ and $r_2 = z_1' > Rez_2' \geq \cdots \geq Rez_{n_2}'$ hold, then the eigenvalues of $D_1 \nabla D_2$ are as follows

$$\lambda_1 = r_1 + n_2,\ z_2,\ z_3,\ \cdots,\ z_{n_1},\ z_2',\ z_3',\ \cdots,\ z_{n_2}',$$
$$\lambda_{n_1+n_2} = r_1 - n_1.$$

Proof Note that the complete product $D_1 \nabla D_2$ of digraphs D_1 and D_2 can be written as $D_1 \nabla D_2 = \overline{\overline{D_1} \oplus \overline{D_2}}$. By lemma 1, we get the spectra of $\overline{D_1}$ and $\overline{D_2}$.

$$\text{Spec}(\overline{D_1}) = \{n_1 - 1 - r_1, -1 - z_2, -1 - z_3, \cdots, -1 - z_{n_1}\},$$

$$\text{Spec}(\overline{D_2}) = \{n_2 - 1 - r_2, -1 - z_2', -1 - z_3', \cdots, -1 - z_{n_2}'\}.$$

Since the spectrum of $\overline{D_1} \oplus \overline{D_2}$ is composed of eigenvalues of $\overline{D_1}$ and $\overline{D_2}$, we obtain $\text{Spec}(\overline{D_1} \oplus \overline{D_2}) = \{n_1 - 1 - r_1, -1 - z_2, -1 - z_3, \cdots, -1 - z_{n_1}, n_2 - 1 - r_2, -1 - z_2', -1 - z_3', \cdots, -1 - z_{n_2}'\}$. Since $D_1 \nabla D_2$ is strongly connected and $(r_1 + n_2)$-regular, we have $\lambda_1 = r_1 + n_2$. By using lemma 1 and $\overline{D_1 \nabla D_2} = \overline{D_1} \oplus \overline{D_2}$, we get

$$\mathrm{Spec}(D_1 \nabla D_2)=\mathrm{Spec}\left(\overline{\overline{D_1} \oplus \overline{D_2}}\right)=\{r_1+n_2, z_2, z_3, \cdots, z_{n_1}, z_2', z_3', \cdots, z_{n_2}', r_1-n_1\}. \quad \square$$

By Theorem 2, the relationship between $E(D_1 \nabla D_2)$ and $E(D_i)$ can be found, where $i = 1, 2$.

Corollary 3. Let D_1 be a strongly connected r_1-regular digraph of order n_1 and D_2 be a strongly connected r_2-regular digraph of order n_2. Suppose $r_1 + n_2 = r_2 + n_1$. Then $E(D_1 \nabla D_2) = E(D_1) + E(D_2) + n_1 + n_2 - r_1 - r_2$.

Proof By Theorem 2, we get

$$\mathrm{Spec}(D_1 \nabla D_2)=\{r_1+n_2, z_2, z_3, \cdots, z_{n_1}, z_2', z_3', \cdots, z_{n_2}', r_1-n_1\}.$$

Thus,

$$\mathrm{E}(D_1 \nabla D_2)=|r_1+n_2| + |Rez_2| + \cdots + |Rez_{n_1}| + |Rez_2'| + \cdots + |Rez_{n_2}'| + |r_1 - n_1|$$

$$=r_1+n_2+\sum_2^{n_1}|Rez_i|+\sum_2^{n_2}|Rez_i'|+n_1-r_1$$

$$=n_1+n_2+E(D_1)-r_1+E(D_2)-r_2. \ \square$$

Corollary 4. Let D_1 be a strongly connected r_1-regular digraph of order n_1 and D_2 be a strongly connected r_2-regular digraph of order n_2. Suppose $r_1 + n_2 = r_2 + n_1$. If $E(D_1) + E(D_2) = n_1 + n_2 + r_1 + r_2 - 2$ holds, then $E(D_1 \nabla D_2)=2(n_1+n_2-1)$.

Corollary 5. Let D_1 be a strongly connected r_1-regular digraph of order n_1 and D_2 be a strongly connected r_2-regular digraph of order n_2. Suppose $n_1 = n_2 = n$, $r_1 = r_2 = r$. If $E(D_1) + E(D_2) = 2\mathrm{n} + 2\mathrm{r} - 2$, then $E(D_1 \nabla D_2)=2(n_1+n_2-1)$.

Example 6. Let $\overrightarrow{C_6}$ be a directed circle of order 6, which is a 1-regular digraph with $E(\overrightarrow{C_6}) = 4$. If D_2 is a $(1+2a)-$regular digraph with $(6+2a)$ vertices and $E(D_2)=8+4a$, where a is a positive integer. Then $E(\overrightarrow{C_6} \nabla D_2)=2(2a+12-1)$.

3. Sequences of Borderenergetic Digraphs

In this section, a sequence of borderenergetic digraphs is constructed. Let $\mathcal{D}^*$ be a set of digraphs, where $\mathcal{D}^* = \{D_0^*, D_1^*, D_2^*, \cdots, D_n^*, \cdots\}$. That is

$$D_1^* = D_0^* \nabla D^0,\ D_2^* = D_1^* \nabla D^0,$$
$$D_3^* = D_2^* \nabla D^0,\ \cdots,\ D_n^* = D_{n-1}^* \nabla D^0, \cdots$$

Theorem 7. Let D_0^* be a strongly connected r_1-regular digraph of order n_1 and D^0 be a strongly connected r_2-regular digraph of order n_2. Suppose $r_1 + n_2 = r_2 + n_1$. If $E(D_0^*) = 2(n_1 - 1)$ and $E(D^0) = 2r_2$, then D_i^* is the digraph whose energy is equal to $2(n(D_i^*) - 1)$ and i is a nonnegative integer, where $\mathrm{n}(D_i^*) = \mathrm{i}\, n_2 + n_1$, and $n(D_i^*)$ is the order of D_i^*.

Proof As $i = n$, we see that D_n^* is a $(r_1 + nn_2)-$regular digraph with $(nn_2 + n_1)$ vertices. By Corollary 3, we have

$$E(D_n^*) = E(D_{n-1}^*) + E(D^0) + 2(n_1 - r_1),$$

$$E(D_{n-1}^*) = E(D_{n-2}^*) + E(D^0) + 2(n_1 - r_1),$$

$$E(D_{n-2}^*) = E(D_{n-3}^*) + E(D^0) + 2(n_1 - r_1),$$

$$\cdots\cdots$$

$$E(D_2^*) = E(D_1^*) + E(D^0) + 2(n_1 - r_1),$$

$$E(D_1^*) = E(D_0^*) + E(D^0) + 2(n_1 - r_1).$$

Thus,

$$E(D_n^*)= E(D_0^*) + nE(D^0) + 2n(n_1 - r_1)$$

$$=2(n_1-1) + 2nr_2 + 2n(n_1 - r_1)$$

$$=2(n_1 + nn_2 - 1) \ \square$$

Example 8. Let $\overrightarrow{C_4}$ be a directed circle with four vertices. Let D_0^* be a 5-regular digraph with eight vertices and $D^0 = \overrightarrow{C_4}$. Let $\mathcal{D}^* = \{D_0^*, D_1^*, D_2^*, \cdots, D_n^*, \cdots\}$, where

$$D_1^* = D_0^* \nabla \overrightarrow{C_4},\ D_2^* = D_1^* \nabla \overrightarrow{C_4},$$
$$D_3^* = D_2^* \nabla \overrightarrow{C_4},\ \cdots,\ D_n^* = D_{n-1}^* \nabla \overrightarrow{C_4}, \cdots$$

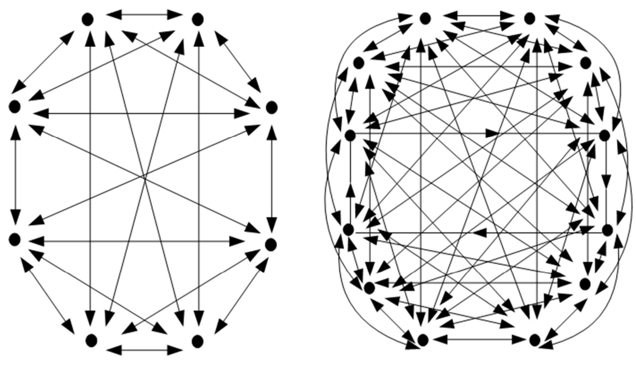
The digraphs D_0^* and D_1^* are given in Figure 3.

As D_n^* is a $(5+4n)$-regular digraph with $(8+4n)$ vertices, $E(D_0^*) = 14$ and $E(\overrightarrow{C_4}) = 2$. By Theorem 7, we have

$$E(D_n^*)= E(D_0^*) + nE(\overrightarrow{C_4}) + 6n$$

$$=2(8-1) + 2n + 6n$$

$$=2(4n+8-1).$$



D_0^* D_1^*

Figure 3. *The digraphs* D_0^* *and* D_1^*.

4. Conclusion

In this paper, we obtain the spectrum of the complete product for two regular digraphs, and construct a class of sequences of digraphs satisfying $E(D) = 2(n-1)$. All digraphs given in this work are regular. For the further study on this topic, the spectra of the digraphs obtained by two irregular digraphs under some graph operations and constructing a borderenergetic digraph can be considered.

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